Robust Blurred Image Recovery Using Minimax and Semi-Definite Programming Approaches

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***Abstract*—In this paper we propose novel approaches for robust blur removal and image reconstruction considering uncertainty in the blur kernel. The stochastic minimization approach models the kernel uncertainty as a stochastic spatial random process, while the worst case reconstruction is a robust minimax scheme which minimizes the maximum image distortion over a set of uncertainty blur kernels. The worst case minimization is demonstrated to be capable of image recovery employing knowledge of the nominal blur kernel matrix and uncertainty strength. Further, we also propose a semi-definite programming (SDP) based scheme for image recovery employing a linear matrix uncertainty model. Another key strength of the proposed schemes is that they can readily incorporate an L_1 norm based total variational cost, thereby resulting in significantly superior performance of image reconstruction. Simulation results show that our approach is able to recover the images with superior visual clarity and lower mean squared-error compared to conventional nominal kernel based non-robust image recovery approaches.**

I. Introduction

Image restoration, which refers to the process of recovering an original image from a distorted version, is a key issue in modern imaging systems and applications. For instance, such blurring distortions typically arise due to motion of the object or atmospheric effects. It has been established in literature that in such scenarios the overall distortion can be reasonably accurately modeled as a linear shift-invariant convolutive kernel operation on the original image followed by the degrading effect of additive noise. The image reconstruction problem is well known to be an ill-conditioned problem since small changes in the original image result in a large distortion at the output or frequently there exists more than one solution to the problem. Image deblurring methods can be broadly classified as blind and non-blind. The former scenario is significantly challenging for image recovery as the blur kernel is unknown. Since the blind image deconvolution problem is ill-posed, having an infinite number of solutions, one has to typically impose additional restrictions on the blur kernel for image recovery. For instance in works such as [1],[2], the blur kernel is restricted to motion and focus blur to circumvent the ill-posed nature of the problem. A more detailed overview of blind image deconvolution can be found in [3].

In non-blind scenarios, even though the blur kernel is known, image recovery is frequently challenging because of the ill-conditioned nature of the problem. There are several works in existing literature on image reconstruction that have studied non-blind deconvolution. In several such works it has been observed to be overwhelmingly difficult to solve the non-blind deblurring problem when the distorted image is additionally corrupted by additive noise. As the blurring operator is ill-conditioned, even noise of modest noise power can strongly contaminate the image signal. This problem is serious when one possesses knowledge of the blur kernel and tends to become worse with increasing uncertainty in the blur kernel. This also one of the central reasons for narrow applicability of non-blind techniques compared to blind ones. The non-blind methods proposed in works such as [4],[5],[6] successfully overcome this problem by utilizing prior information regarding the image to be recovered in a Bayesian or maximum a posteriori (MAP) framework.

Non-blind deblurring methods are extremely sensitive to mismatches between the estimated and actual blur kernel. However, frequently in practice the blur kernel is known only to a degree of uncertainty in the kernel coefficients. In such scenarios, conventional image recovery approaches based on exact knowledge of the blurring kernel can lead to significant image distortion arising out of kernel mismatches. Recent path breaking advances in the field of convex optimization [7] have led to the development of sophisticated optimization techniques for robust signal processing. These techniques are based on the minimax and semi-definite programming criterion and fast interior point methods for optimal solution computation. Hence, in this context we propose robust image recovery approaches which are tolerant of the natural uncertainty in the blur kernel matrix thereby reducing the sensitivity of the solution to the accuracy of the kernel estimate, thus resulting in superior image reconstruction over a wide range of blur kernels belonging to an appropriate kernel uncertainty set. Hence our scheme frames the image recovery problem as an intermediate problem between blind and non-blind scenarios. It considers the frequent availability of limited kernel knowledge in practice, while acknowledging the realities of inaccuracies in kernel estimation. In our work we consider a nominal knowledge of the blur kernel around which the blur kernel uncertainty set is concentrated. Further, we consider several plausible models for uncertainty set modeling. We introduced a statistical approach such as stochastic minimization, where the kernel uncertainty is modeled as a spatial stochastic random process. Subsequently, the proposed worst case image distortion minimization framework minimizes the maximum possible degradation of the recovered image over a large uncertainty set, thus ensuring robustness of the reconstruction process. Further, we also present a novel semi-definite programming approach for image reconstruction based on a linear matrix kernel uncertainty formulation. Further, we also consider L_1 norm based regularization to significantly improve the performance of the above image reconstruction techniques. All the proposed schemes are based on the recently developed paradigm of convex optimization, which has been shown to result in significantly enhanced signal processing refinements compared to conventional techniques. The development of several power convex solvers recently has thus resulted in developing powerful techniques for signal processing, specifically in the case of image reconstruction. We compare the performance of the proposed robust image reconstruction techniques to the conventional uncertainty agnostic fixed kernel schemes employing mean squared-error and visual quality metrics. It is seen from the results therein that the proposed robust schemes significantly enhance the quality of image reconstruction.

The rest of the paper is organized as follows. Section II next discusses the image distortion and reconstruction model. Section III presents stochastic minimization based robust recovery while sections IV, V present the minimax and SDP based worst care error minimization approaches respectively. Section VI contains the simulation results and we present our conclusions in section VII.

II. Image Reconstruction Model

An image can be represented by the two dimensional light intensity function $f(x,y)$ which corresponds to the intensity at the spatial coordinates (x,y). Consider an observed distorted image signal $g(x,y)$ obtained as the result of a blurring and additive noise degradation of the image of interest $f(x,y)$. It has been established in literature that the blurring operation can be well approximated by a linear spatially invariant convolutive filter. Image restoration essentially aims to recover a reconstructed image signal $\hat{f}\,(x,y)$ that is a close approximation of the original image signal $f(x,y)$, or in other words, to remove the effect of the distorting kernel. Let the blur kernel be represented by the 2D filter $h\,(x,y)$. Hence, as a consequence of the linearity of the distortion process, the output signal $g\,(x,y)$ for the described image blur model can expressed as,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,(\alpha,\beta)\,h\,(x-\alpha,y-\beta)\;d\alpha\,d\beta+\eta(x,y),$$

where $\eta(x,y)$ is an additive spatially white noise process. The above signal $g\,(x,y)$ can be suitably sampled to obtain an $M\times N$ digital image represented by the spatial intensity matrix $\mathbf{G}\in\mathbb{R}^{M\times N}$. Hence, the above system model for image distortion can be represented by the discrete filter output,

$$\mathbf{G}\,(m,n)=\sum_{\tilde{m}=0}^{M-1}\sum_{\tilde{n}=0}^{N-1}\mathbf{F}\,(\tilde{m},\tilde{n})\,\mathbf{H}\,(m-\tilde{m},n-\tilde{n})\,.$$

The image matrices $\mathbf{G},\mathbf{F}$ can be conveniently represented as vectors by stacking their columns in lexicographic order. Let these MN dimensional input and output image vectors be represented by $\mathbf{f}$ and $\mathbf{g}$ respectively. The system model for the observed distorted image is given as,

$$\mathbf{g}=\mathbf{H}\mathbf{f}+\eta, \tag{1}$$

where the convolution matrix $\mathbf{H}$ is the blur kernel and has a block toeplitz with toeplitz blocks (BTTB) structure given as,

$$\tilde{\mathbf{H}}=\begin{bmatrix}\tilde{\mathbf{H}}_0 & \tilde{\mathbf{H}}_1 & \tilde{\mathbf{H}}_2 & \dots & \tilde{\mathbf{H}}_M\\ \tilde{\mathbf{H}}_{-1} & \tilde{\mathbf{H}}_0 & \tilde{\mathbf{H}}_1 & \dots & \tilde{\mathbf{H}}_{M-1}\\ \tilde{\mathbf{H}}_{-2} & \tilde{\mathbf{H}}_{-1} & \tilde{\mathbf{H}}_0 & \dots & \tilde{\mathbf{H}}_{M-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \tilde{\mathbf{H}}_{-N} & \tilde{\mathbf{H}}_{-N+1} & \tilde{\mathbf{H}}_{-N+2} & \dots & \tilde{\mathbf{H}}_0\end{bmatrix}.$$

The above linear convolution problem can be closely approximated as a circular convolution problem through appropriate zero padding for mathematical tractability. Assuming the standard periodic boundaries criterion, the corresponding kernel $\mathbf{H}$ possesses a block circulant with circulant block matrices structure (BCCB) [8] described as,

$$\mathbf{H}=\begin{bmatrix}\mathbf{H}_0 & \mathbf{H}_1 & \mathbf{H}_2 & \dots & \mathbf{H}_M\\ \mathbf{H}_M & \mathbf{H}_0 & \mathbf{H}_1 & \dots & \mathbf{H}_{M-1}\\ \mathbf{H}_{M-1} & \mathbf{H}_M & \mathbf{H}_0 & \dots & \mathbf{H}_{M-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_M & \dots & \mathbf{H}_0\end{bmatrix}.$$

The block rows of $\mathbf{H}$ are circulant shifts of each other and each block element contains a similar circulant coefficient structure. The significance of this form is that there exist extremely computationally efficient approaches to solve the above image recovery problem, rendering it mathematically tractable even for large image sizes. This property of $\mathbf{H}$ arises from its space invariance property [8]. A general 2-Dimensional BCCB matrix can be diagonalized using the 2-Dimensional unitary discrete Fourier transform matrix i.e,

$$\mathbf{H}=\mathbf{F}^H\mathbf{\Lambda}\mathbf{F}, \tag{2}$$

where $\mathbf{F}=\mathbf{F}_1\otimes\mathbf{F}_1$ and $\mathbf{F}_1$ is the 1-Dimensional unitary discrete Fourier transform matrix. The matrix $\mathbf{\Lambda}$ is a diagonal matrix whose elements are the Fourier coefficients of the spatial 2-D blur kernel $\mathbf{H}$. The blur kernel can be modeled as comprising of independent horizontal and vertical blur filters with effective convolution matrix $\mathbf{H}$ given as,

$$\mathbf{H}=\mathbf{H}_v\mathbf{H}_h,$$

where $\mathbf{H}_v,\mathbf{H}_h$ are the vertical and horizontal distortion operators respectively. Both of these matrices possess the BCCB structure, and hence the product matrix $\mathbf{H}$ also retains the BCCB structure. The image restoration problem given the distortion model in (1) can then be modeled by the least-squares cost minimization based optimization problem, which is known to yield the maximum likelihood (ML) estimate of the original image vector. Hence, given the blur kernel $\mathbf{H}$, the optimal least error reconstructed image $\mathbf{f}$ is given as,

$$\hat{\mathbf{f}} = \arg\min \ \|\mathbf{H}\mathbf{f} - \mathbf{g}\|_2^2 .$$

This problem can be readily solved to yield the closed form solution for the least squares or nominally restored image vector $\hat{\mathbf{f}} = \mathbf{H}^{\dagger}\mathbf{g}$, where $\mathbf{H}^{\dagger}$ denotes the Moore-Penrose pseudo inverse of the kernel $\mathbf{H}$. Moreover, as has been gaining popularity in recent times, in order to impose the smoothness constraints widely found in naturally occurring images, one can minimize the total variation of the reconstructed image vector $\hat{\mathbf{f}}$ through regularization by the L_1 cost function. For this purpose we can define the first order finite difference operators along the horizontal and vertical directions $\mathbf{D}_x$ and $\mathbf{D}_y$ respectively as,

$$\begin{aligned} \mathbf{D}_x &= \mathrm{vec}\left(f(x+1,y) - f(x,y)\right) \\ \mathbf{D}_y &= \mathrm{vec}\left(f(x,y+1) - f(x,y)\right). \end{aligned}$$

III. Robust Image Recovery With Kernel Uncertainty

A significant shortcoming of the above standard least squares based framework for image reconstruction is the requirement of precise knowledge of the blur kernel $\mathbf{H}$. However, frequently in practice it is extremely challenging to precisely determine the blur kernel, as a result of which it is known only up to a degree of uncertainty. In this context a more realistic model for the blur kernel is obtained through an uncertainty variation in the nominal blur kernel $\mathbf{H}$. Hence, the true blur kernel $\bar{\mathbf{H}} \in \mathbb{R}^{MN \times MN}$ can be efficiently modeled as a random matrix with mean $\mathbf{H}$ given as,

$$\bar{\mathbf{H}} = \mathbf{H} + \mathbf{U},$$

where $\mathbf{U}$ describes the zero-mean random uncertainty matrix. Let uncertainties in the horizontal and vertical blur kernels be denoted by $\mathbf{A}$ and $\mathbf{B}$ respectively, which comprise of zero mean and independent random uncertainty coefficients. Hence, the net uncertainty blur kernel can be approximated as,

$$\bar{\mathbf{H}} = (\mathbf{H}_v + \mathbf{A})(\mathbf{H}_h + \mathbf{B}) \approx \mathbf{H} + \mathbf{H}_v\mathbf{B} + \mathbf{A}\mathbf{H}_h,$$

where the last approximation follows from ignoring the second order error matrix product $\mathbf{AB}$. The robust image recovery problem with kernel uncertainty can be recast as,

$$\min \left\| \underbrace{(\mathbf{H} + \mathbf{H}_v\mathbf{B} + \mathbf{A}\mathbf{H}_h)}_{\bar{\mathbf{H}}} \mathbf{f} - \mathbf{g} \right\|_2^2 .$$

Next we describe various robust approaches such as stochastic and worst case error minimization for image recovery and distortion minimization.

A. Stochastic Minimization Based Robust Recovery

The uncertainty matrix $\mathbf{U}$ can be modeled as a zero-mean stochastic spatial random process possessing a BCCB filter structure, with the filter entries derived as independent random coefficients of uncertainty variance σ_u^2. Hence, the stochastic error minimization based optimization criterion for robust image recovery can be modeled by the convex optimization paradigm [7] described as,

$$\min \ \mathrm{E}\left\{ \left\|\bar{\mathbf{H}}\mathbf{f} - \mathbf{g}\right\|_2^2 \right\}.$$

The above objective function for reconstruction of the image vector $\mathbf{f}$ can be further simplified as,

$$\begin{aligned} \mathrm{E}\left\{ \left\|\bar{\mathbf{H}}\mathbf{f} - \mathbf{g}\right\|_2^2 \right\} &= \mathrm{E}\left\{ (\mathbf{H}\mathbf{f} - \mathbf{g} + \mathbf{U}\mathbf{f})^H (\mathbf{H}\mathbf{f} - \mathbf{g} + \mathbf{U}\mathbf{f}) \right\} \\ &= \|\mathbf{H}\mathbf{f} - \mathbf{g}\|_2^2 + \mathbf{f}^H \mathrm{E}\left\{\mathbf{U}'\mathbf{U}\right\} \mathbf{f}, \end{aligned}$$

where the result $\mathrm{E}\left\{ (\mathbf{U}\mathbf{f})^H (\mathbf{H}\mathbf{f} - \mathbf{g}) \right\} = 0$ has been employed in the above simplification. Let the matrix $\mathbf{P}$ be defined as the covariance $\mathbf{P} = \mathrm{E}\left\{\mathbf{U}^H\mathbf{U}\right\}$. The above cost function for robust image recovery can thus be equivalently represented as,

$$\|\mathbf{H}\mathbf{f} - \mathbf{g}\|_2^2 + \mathbf{f}^H\mathbf{P}\mathbf{f}, \tag{3}$$

where $\mathbf{P}$ is fixed for a given uncertainty variance. From the BCCB structure of the uncertainty matrix $\mathbf{U}$ described above, the uncertainty covariance $\mathbf{P}$ can be simplified in terms of the matrices $\mathbf{H}_v, \mathbf{H}_h$ as,

$$\begin{aligned} \mathbf{P} &= \mathrm{E}\left\{ \underbrace{(\mathbf{H}_v\mathbf{B} + \mathbf{A}\mathbf{H}_h)^H}_{\mathbf{U}^H} \underbrace{(\mathbf{H}_v\mathbf{B} + \mathbf{A}\mathbf{H}_h)}_{\mathbf{U}} \right\} \\ &= \mathrm{E}\left\{\mathbf{B}^H\mathbf{H}_v^H\mathbf{H}_v\mathbf{B}\right\} + \mathrm{E}\left\{\mathbf{H}_h^H\mathbf{A}^H\mathbf{A}\mathbf{H}_h\right\} \end{aligned}$$

where the simplification employs the assumption that the uncertainty matrices $\mathbf{A}, \mathbf{B}$ comprise of independent zero-mean noise components. It can be observed that the above optimization problem possesses the Tikhonov regularized least squares problem structure and hence a closed form solution for image reconstruction is given as,

$$\hat{\mathbf{f}} = \left(\mathbf{H}^H\mathbf{H} + \mathbf{P}\right)^{-1} \mathbf{H}^H\mathbf{g}. \tag{4}$$

An very low computational complexity scheme for implementing the above image restoration algorithm can be developed employing the BCCB structure of the component kernel matrices. This can be derived as,

$$\begin{aligned} \hat{\mathbf{f}} &= \left(\left(\mathbf{F}^H\mathbf{\Lambda}\mathbf{F}\right)^H \left(\mathbf{F}^H\mathbf{\Lambda}\mathbf{F}\right) + \mathbf{P} \right)^{-1} \mathbf{H}^H\mathbf{g} \\ &= \left(\mathbf{F}^H\mathbf{\Lambda}^2\mathbf{F} + \mathbf{P}\right)^{-1} \left(\mathbf{F}^H\mathbf{\Lambda}\mathbf{F}\right)\mathbf{g} \\ &= \mathbf{F}^H\left(\mathbf{\Lambda}^2 + \mathbf{F}\mathbf{P}\mathbf{F}^H\right)^{-1} \mathbf{\Lambda}\mathbf{F}\mathbf{g} \\ &= \mathbf{F}^H\left(\mathbf{\Lambda}^2 + \mathbf{\Lambda}_p\right)^{-1} \mathbf{\Lambda}\mathbf{F}\mathbf{g}, \end{aligned}$$

where $\mathbf{\Lambda}_p = \mathbf{F}\mathbf{P}\mathbf{F}^H$ is obtained from the Fourier decomposition of the kernel uncertainty covariance matrix $\mathbf{P}$. The inverse of the inner diagonal matrix can be computed in a straight forward fashion, thus drastically reducing the complexity of the matrix inversion for the native expression in (4). Next we present the minimax scheme for robust image reconstruction.

IV. Minimax Based Robust Recovery

Recently, uncertainty set based worst case distortion minimization approaches have gained a significant research interest in signal processing. Consider matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{MN \times MN}$ as the fixed uncertainty basis matrices. Any blur matrix $\bar{\mathbf{H}}$ belonging to an uncertainty set $\mathcal{H}$ can be simplified as,

$$\bar{\mathbf{H}} \quad = \quad (\mathbf{H}_v + u_1 \mathbf{A}) \mathbf{H}_h + u_2 \mathbf{B} = \mathbf{H} + u_2 \mathbf{H}_v \mathbf{B} + u_1 \mathbf{A} \mathbf{H}_h$$

where the quantities $u_1, u_2 \in [0, 1]$ denote the bounded unknown uncertainty factors. Hence, the bounded and nonempty uncertainty set $\mathcal{H} \subset \mathbb{R}^{MN \times MN}$ for the kernel matrix $\bar{\mathbf{H}}$ can be defined as,

$$\mathcal{H} = \{\mathbf{H} + u_1 \mathbf{A} + u_2 \mathbf{B} | u_1, u_2 \in [-1, 1]\},$$

The optimization problem for worst case error minimization over the kernel uncertainty set $\mathcal{H}$ can be readily derived as,

$$\min \max \left\| \bar{\mathbf{H}} \mathbf{f} - \mathbf{g} \right\|_2^2 \text{ s.t. } \bar{\mathbf{H}} \in \mathcal{H}. \tag{5}$$

The above optimization can be readily seen to be convex in nature as the $\max$ function is convex [7]. Below we present a scheme to simplify the above optimization problem. Given the structure of the uncertainty set $\mathcal{H}$ described above, it can be readily seen that the worst case cost function in (5) above is second order convex in the parameters u_1, u_2. Hence, the worst case error occurs at the end points of the uncertainty intervals corresponding to $u_1, u_2 \in \{-1, 1\}$. The above cost function for worst case error minimization can be recast as,

$$\min \max \left\| (\mathbf{H} \pm \mathbf{H}_v \mathbf{B} \pm \mathbf{A} \mathbf{H}_h) \mathbf{f} - \mathbf{g} \right\|_2^2 \tag{6}$$

Similar to the stochastic minimization paradigm, the BCCB properties of the constituent matrices can be employed to drastically reduce the complexity of the above optimization problem. As both the matrices $\mathbf{H}_v$ and $\mathbf{H}_h$ possess the BCCB structure these can be diagonalized as, $\mathbf{H}_v = \mathbf{F}^H \mathbf{\Lambda}_v \mathbf{F}$ and $\mathbf{H}_h = \mathbf{F}^H \mathbf{\Lambda}_h \mathbf{F}$. Employing the fact that the Fourier matrix $\mathbf{F}$ is unitary i.e. $\mathbf{F}\mathbf{F}^H = \mathbf{F}^H \mathbf{F} = \mathbf{I}$, the cost function for minimax based robust image recovery can be simplified as,

$$\min \max \left\| (\mathbf{\Lambda} \pm \mathbf{\Lambda}_v \pm \mathbf{\Lambda}_h) \bar{\mathbf{f}} - \bar{\mathbf{g}} \right\|_2^2,$$

where $\bar{\mathbf{f}} = \mathbf{F}\mathbf{f}$ and $\bar{\mathbf{g}} = \mathbf{F}\mathbf{g}$. The above optimization problem can be solved in a straightforward fashion. The inverse Fourier transform of the solution $\bar{\mathbf{f}}$ can be computed to obtain the reconstructed image vector $\mathbf{f}$ as $\mathbf{f} = \mathbf{F}^{-1} \bar{\mathbf{f}}$. Further, total variation TV_1 constraint based image reconstruction approaches have attracted significant attention in recent times because of their superior performance. Hence, one can enhance the performance of the worst case robust image recovery above by including an L_1 regularization component to modify the optimization objective as,

$$\min \max \; \left\| \mathbf{H}\mathbf{f} - \mathbf{g} \right\|_2^2 + \left\| \mathbf{D}_x \mathbf{f} \right\|_1 + \left\| \mathbf{D}_y \mathbf{f} \right\|_1.$$

The solution of the above convex optimization problem yields the L_1 regularization based reconstructed image vector.

V. Norm Bounded Linear Matrix Uncertainty

A more sophisticated and practical model for the kernel uncertainty is given by the linear matrix uncertainty model. As per this model, the uncertain kernel matrix $\bar{\mathbf{H}}$ can be described as,

$$\bar{\mathbf{H}} = \mathbf{H} + u_1 \mathbf{H}_1 + u_2 \mathbf{H}_2 + \ldots + u_p \mathbf{H}_p \text{ s.t. } \|\mathbf{u}\| \leq 1, \tag{7}$$

where $\mathbf{u} = [u_1, u_2, ..., u_p]^T$ is the norm bound uncertainty coefficient vector. The worst case error can be defined as,

$$\begin{aligned} & \sup_{\|u\| \leq 1} \left\| (\mathbf{H} + u_1 \mathbf{H}_1 + u_2 \mathbf{H}_2 + \; + u_p \mathbf{H}_p) \mathbf{f} - \mathbf{g} \right\| \\ = & \sup_{\|u\| \leq 1} \left\| \mathbf{P}(\mathbf{f}) u + \mathbf{J}(\mathbf{f}) \right\| \end{aligned}$$

where the matrices $\mathbf{P}(\mathbf{f}) \in \mathbb{R}^{MN \times p}$ and $\mathbf{J}(\mathbf{f}) \in \mathbb{R}^{MN \times 1}$ are defined as, $\mathbf{P}(\mathbf{f}) \triangleq [\mathbf{H}_1 \mathbf{f}, \; \mathbf{H}_2 \mathbf{f}, \; \ldots, \mathbf{H}_p \mathbf{f}]$ and $G(\mathbf{f}) \triangleq \mathbf{H}\mathbf{f} - \mathbf{g}$. Hence the above optimization problem for linear matrix uncertainty based image recovery can be recast as,

$$\min . \max_{\|u\| <= 1} \left\| \mathbf{P}(\mathbf{f}) \mathbf{u} + G(\mathbf{f}) \right\|_2^2.$$

In [7] it has been demonstrated that the Lagrange dual of above optimization problem can be simplified as the following semi-definite programming (SDP) based optimization problem as,

$$\begin{aligned} \min . & \quad (t + \lambda) \\ \text{s.t.} & \quad \mathbf{K} \succeq 0 \end{aligned}$$

where the matrix $\mathbf{K}$ is defined as,

$$\mathbf{K} = \begin{bmatrix} \mathbf{I} & \mathbf{P}(\mathbf{f}) & \mathbf{G}(\mathbf{f}) \\ F(\mathbf{f})^T & \lambda \mathbf{I} & \mathbf{0} \\ G(\mathbf{f})^T & \mathbf{0} & t \end{bmatrix}.$$

The $\succeq$ in the above optimization problem represents the generalized matrix inequality on the cone of positive semi-definite matrices. The above optimization problem can be conveniently solved employing convex solvers such as CVX. Below we present simulation results to illustrate the performance of the above image reconstruction algorithms and compare their performance with that of the uncertainty agnostic nominal kernel based reconstruction approaches.

VI. Simulations

We applied the above described robust reconstruction approaches to an $M = N = 48$ sized image data set i.e. comprising of 48×48 pixel sized images as shown in Fig.3 The image recovery was performed employing the robust schemes presented above and the results were compared with those corresponding to the nominal restored ones. In Fig.1 we plot the average mean square-error (MSE) between the original and recovered image vectors $\mathbf{f}$ and $\hat{\mathbf{f}}$ respectively for the stochastic minimization based robust recovery and the nominal estimate based conventional approach. The blur matrix is derived from a 6×6 blur kernel and the image signal to noise power ratio after corruption by additive Gaussian noise is at an SNR level of 30dB. The reconstruction is computed for 100 iterations for each uncertainty level. It can be observed from the figure

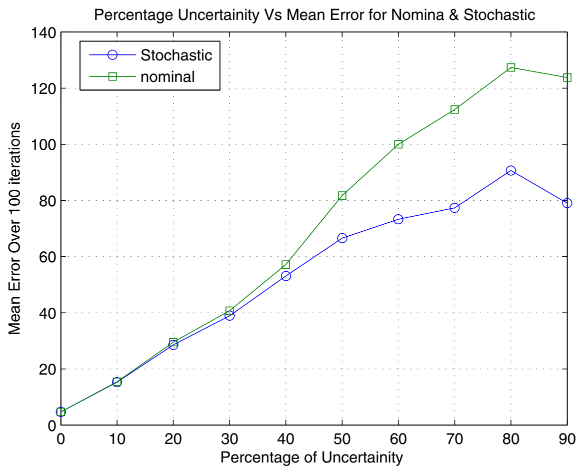

Fig. 1. MSE vs. Percentage Uncertainty for Nominal & Stochastic Restoration

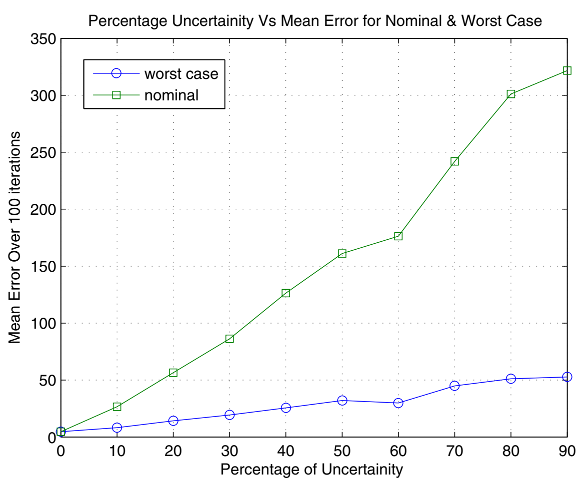

Fig. 2. MSE vs. Percentage Uncertainty for Nominal & Worst Case Restoration

that the MSE of the recovered image is significantly lower for the stochastic approach compared to the mean kernel based conventional approach.

In Fig.2 we compared the average MSE for the worst case and nominal approaches. Similar to the previous scenario, a 6×6 blur kernel is used to construct the blur matrix $\mathbf{H}$ and the blurred image is corrupted with additive noise of SNR 30dB. It can be seen that the average MSE of the worst case solution based minimax reconstruction is flat and doest not increase with increasing uncertainty compared to the average MSE of the nominal solution, which progressively increases with the uncertainty level. Thus, it is evident that while the nominal approach performs well when precise knowledge of the kernel matrix is available, it is sensitive to kernel mismatch error, leading to a substantial increase in the MSE of reconstruction with increasing uncertainty.

In Fig.4 we compare the visual quality of image reconstruction of the stochastic and nominal approaches. In

Fig. 3. Standard Gray Scale Test Images used in simulations

Fig. 4. Comparison of Reconstructed images for Stochastic and Nominal approaches. From left to right in each row blurred - corrupted - stochastic recovery - nominal recovery.

Fig.5, we present a similar comparison for the worst case and conventional approaches. An uncertainty level of 60% is considered in the blur matrix. The reconstruction image quality clearly shows that the worst case as well as stochastic approaches perform better compared to the nominal one. All the simulations are carried out using the convex optimization tool CVX [9]. Fig.6 shows the reconstructed images employing the worst case minimax approach with and without the TV_1 regularization constraint. An uncertainty level of 30% is considered in the blur kernel. It can be observed that the recovered image quality is sharper as image edges are well preserved with TV_1 regularization. Fig.7 illustrates the sensitivity of the worst case and nominal solutions to the changes in the uncertainty parameter u. In this test problem we modeled uncertainty as a norm ball and solved the problem using the SDP formulation described in section V. To show the sensitivity of these approximate solutions we generated 10^4 parameter vectors uniformly distributed on the unit disk and evaluated the residues $\|\mathbf{H} + u_1\mathbf{H}_1 + u_2\mathbf{H}_2\|_2^2$. For ease of implementation we considered blur in the vertical direction only and the approach can be readily extended to include the horizontal direction also. From the graph we infer that the residuals of the least-squares solution are widely spread compared to the residuals of the worst case solution. Hence we can conclude that the SDP based worst case solution is less sensitive and more stable compared to competing approaches. Further, the stochastic and worst case minimax (finite uncertainty basis based modeling) solutions are also less sensitive to the parameter changes. Also, when CVX was em-

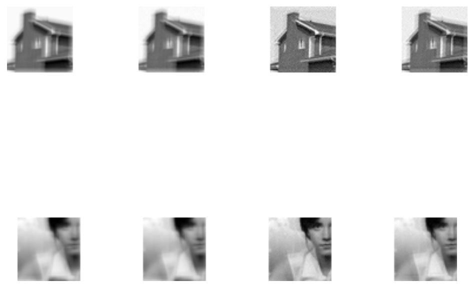

Fig. 5. Comparison of Reconstructed images for Worst Case and Nominal approaches. From left to right in each row blurred - corrupted - worst case(finetset case) recovery - nominal recovery

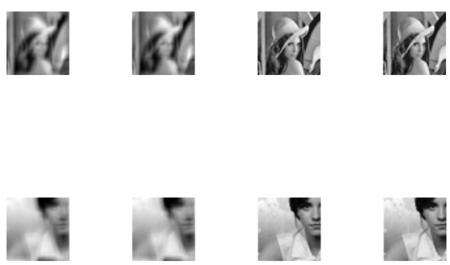

Fig. 6. Image Reconstruction with and without TV_1 regularization in worst case scenario. From left to right blurred - corrupted - recovery with TV_1 - recovery with out TV_1.

ployed for convex optimization with the BCCB structure based Fourier diagonalization, the optimization solver converges to the solution with significantly fewer iterations compared to the unsimplified problem.

VII. Conclusion

In this paper we proposed novel stochastic minimization, worst case and semi-definite programming approaches for uncertainty based robust image reconstruction, motivated by the fact that conventional nominal kernel based approaches are sensitive to kernel mismatches arising out of estimation errors. The kernel uncertainty was variously modeled as a stochastic spatial random process, bounded uncertainty set based and as a linear matrix bounded norm uncertainty. The proposed techniques employ the recently developed advancements in the theory of convex optimization. Further, a TV_1 based regularization cost can be readily incorporated into the proposed framework, thus imposing sparsity based image reconstruction constraints and enhancing the image recovery performance. Simulation results demonstrate that the proposed robust image recovery schemes have a superior performance compared to

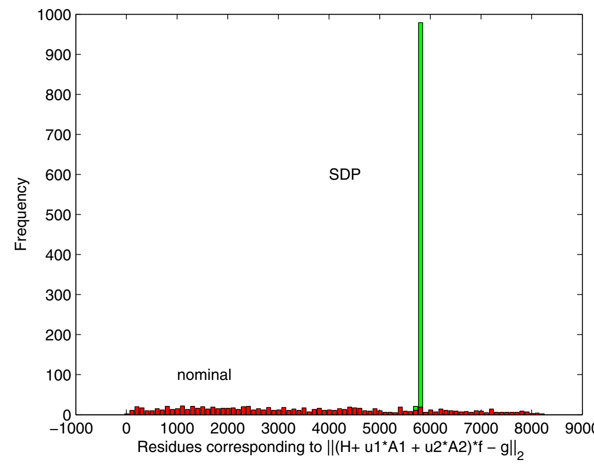

Fig. 7. Residue distribution of worst case and nominal image reconstruction.

conventional techniques.

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